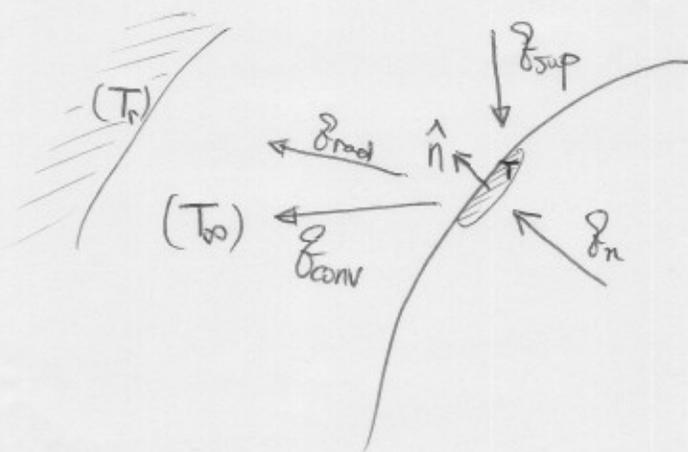


## 1.5. Boundary Conditions

Boundary Conditions specify the temperature ( $T$ ) and/or the heat flow ( $\vec{q}$ ) at the boundaries.

### \* General Boundary Conditions

Energy balance at the surface of a solid.



- $q_{sup}$ : energy supplied to the surface from external source
- $q_{conv} = h(T - T_0)$ : heat loss from surface by convection into an ambient at temperature  $T_0$ .
- $q_{rad} = \epsilon\sigma(T^4 - T_r^4)$ : heat loss from surface by radiation into an ambient at temperature  $T_r$ .
- $q_n = -k \frac{\partial T}{\partial n}$ : conduction heat flux normal to the surface.

$$\text{(heat supply)} \quad q_n + q_{sup} = q_{conv} + q_{rad} \quad \text{(heat loss)}$$

$$-k \frac{\partial T}{\partial n} + q_{sup} = h(T - T_0) + \epsilon\sigma(T^4 - T_r^4)$$

General Boundary Condition:

$$\left| k \frac{\partial T}{\partial n} + hT + \epsilon \sigma T^4 = hT_{\infty} + \rho_{\text{sup}} + \epsilon \sigma T_r^4 \right|$$

at the surface!

\* Boundary Condition of the 1st Kind

$$T|_s = f(\vec{r}, t) \quad \text{temperature distribution is given at boundary}$$

special case:  $\underline{T|_s = 0} \rightarrow$  homogeneous boundary condition of the 1st kind

\* Boundary Condition of the 2nd Kind

$$k \frac{\partial T}{\partial n} \Big|_s = f(\vec{r}, t) \quad \text{heat flux is given at the boundary}$$

special case:  $\underline{\frac{\partial T}{\partial n} \Big|_s = 0} \rightarrow$  homogeneous boundary condition of the 2nd kind

\* Boundary Condition of the 3rd Kind

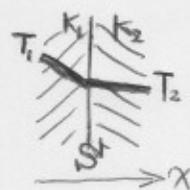
$$k \frac{\partial T}{\partial n} \Big|_s + hT|_s = hT_{\infty}(\vec{r}, t) \quad \text{convection boundary condition}$$

special case:  $\underline{k \frac{\partial T}{\partial n} \Big|_s + hT|_s = 0} \rightarrow$  homogeneous boundary condition of the 3rd kind

\* Interface Boundary Condition:

assuming perfect thermal contact:

$$\left\{ \begin{array}{l} T_1|_s = T_2|_s \\ -k_1 \frac{\partial T}{\partial x} \Big|_s = -k_2 \frac{\partial T}{\partial x} \Big|_s \end{array} \right.$$



## 1.6. Nonhomogeneous Problems

\* General heat conduction Equation

$$\nabla^2 T(\vec{r}, t) + \frac{1}{\kappa} g(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T(\vec{r}, t)}{\partial t}$$

{ Nonhomogeneous if  $g(\vec{r}, t) \neq 0$

{ Homogeneous if  $g(\vec{r}, t) = 0$

\* General Boundary Conditions. — convection

$$k_i \frac{\partial T}{\partial n_i} \Big|_{S_i} + h_i T \Big|_{S_i} = f_i(\vec{r}, t) \quad (\text{at boundary surface } S_i)$$

{ Nonhomogeneous if  $f_i(\vec{r}, t) \neq 0$

{ Homogeneous if  $f_i(\vec{r}, t) = 0$

\* Solving Nonhomogeneous Problems

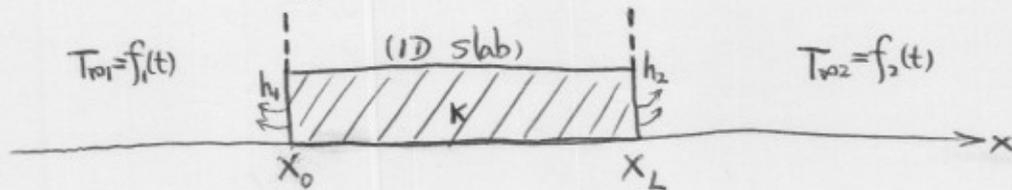
To solve nonhomogeneous equation with nonhomogeneous boundary conditions — Split the problem into a set of simpler problems:

{ Nonhomogeneous equation with homogeneous boundary conditions

{ Homogeneous equation with nonhomogeneous boundary conditions

(usually solvable with "Separation of Variables" method)

\* Example: One-dimensional transient heat conduction of a slab, with heat generation and convection boundary conditions.



Nonhomogeneous equation with nonhomogeneous boundary conditions.

$$\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{1}{k} g(x,t) = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \quad \leftarrow \text{nonhomogeneous}$$

$$\text{B.C.} \quad \begin{cases} -k \frac{\partial T}{\partial x} \Big|_{x=0} + h_1 T \Big|_{x=0} = h_1 f_1(t) \\ k \frac{\partial T}{\partial x} \Big|_{x=L} + h_2 T \Big|_{x=L} = h_2 f_2(t) \end{cases} \quad \leftarrow \text{nonhomogeneous}$$

$$\text{I.C.} \quad T \Big|_{t=0} = F(x) \quad \leftarrow \text{nonhomogeneous}$$

To split this problem into multiple simpler problems, assume  $T(x,t)$  having the following form:

$$T(x,t) = \theta(x,t) + \phi_1(x) f_1(t) + \phi_2(x) f_2(t)$$

( $\theta(x,t)$ ,  $\phi_1(x)$  and  $\phi_2(x)$  satisfy new equations and boundary conditions — simpler problems)

(1) Let  $\phi_1(x)$  be the solution of the following problem:

$$\boxed{\begin{array}{l} \frac{d^2 \phi_1(x)}{dx^2} = 0 \\ \text{B.C. } \int \left\{ \begin{array}{l} -k \frac{d\phi_1}{dx} \Big|_{x=x_0} + h_1 \phi_1 \Big|_{x=x_0} = h_1 \\ k \frac{d\phi_1}{dx} \Big|_{x=x_L} + h_2 \phi_1 \Big|_{x=x_L} = 0 \end{array} \right. \end{array}}$$

← homogeneous equation  
← nonhomogeneous B.C.  
← homogeneous B.C.

(2) Let  $\phi_2(x)$  be the solution of the following problem:

$$\boxed{\begin{array}{l} \frac{d^2 \phi_2(x)}{dx^2} = 0 \\ \text{B.C. } \int \left\{ \begin{array}{l} -k \frac{d\phi_2}{dx} \Big|_{x=x_0} + h_1 \phi_2 \Big|_{x=x_0} = 0 \\ k \frac{d\phi_2}{dx} \Big|_{x=x_L} + h_2 \phi_2 \Big|_{x=x_L} = h_2 \end{array} \right. \end{array}}$$

← homogeneous equation  
← homogeneous B.C.  
← nonhomogeneous B.C.

(3) Determine the equation and boundary/initial conditions for  $\theta(x,t)$  (in order for the splitting to be valid).

Step 1: substitute  $T(x,t) = \theta(x,t) + \phi_1(x)f_1(t) + \phi_2(x)f_2(t)$  into original nonhomogeneous equation.

$$\left[ \frac{\partial^2 \theta(x,t)}{\partial x^2} + \underbrace{\frac{d^2 \phi_1(x)}{dx^2}}_0 f_1(t) + \underbrace{\frac{d^2 \phi_2(x)}{dx^2}}_0 f_2(t) \right] + \frac{1}{k} g(x,t) = \left[ \frac{1}{\alpha} \frac{\partial \theta(x,t)}{\partial t} + \frac{1}{\alpha} \phi_1(x) \frac{df_1(t)}{dt} + \frac{1}{\alpha} \phi_2(x) \frac{df_2(t)}{dt} \right]$$

$$\frac{\partial^2 \theta(x,t)}{\partial x^2} + \frac{1}{k} g(x,t) = \frac{1}{\alpha} \frac{\partial \theta(x,t)}{\partial t} + \frac{1}{\alpha} \left[ \phi_1(x) \frac{df_1(t)}{dt} + \phi_2(x) \frac{df_2(t)}{dt} \right]$$

$$\frac{\partial^2 \theta(x,t)}{\partial x^2} + \underbrace{\left[ \frac{1}{k} g(x,t) - \frac{1}{\alpha} \left[ \phi_1(x) \frac{df_1(t)}{dt} + \phi_2(x) \frac{df_2(t)}{dt} \right] \right]}_{g^*(x,t)} = \frac{1}{\alpha} \frac{\partial \theta(x,t)}{\partial t}$$

Step 2: Substitute  $T(x, t) = \theta(x, t) + \phi_1(x)f_1(t) + \phi_2(x)f_2(t)$   
into original nonhomogeneous boundary conditions.

$$\left\{ \begin{array}{l} \left[ -k \frac{\partial \theta}{\partial x} \Big|_{x=x_0} - k \frac{d\phi_1}{dx} \Big|_{x=x_0} f_1(t) - k \frac{d\phi_2}{dx} \Big|_{x=x_0} f_2(t) \right] + \left[ h_1 \theta \Big|_{x=x_0} + h_1 \phi_1 \Big|_{x=x_0} f_1(t) + h_1 \phi_2 \Big|_{x=x_0} f_2(t) \right] = h_1 f_1(t) \\ \left[ k \frac{\partial \theta}{\partial x} \Big|_{x=x_L} + k \frac{d\phi_1}{dx} \Big|_{x=x_L} f_1(t) + k \frac{d\phi_2}{dx} \Big|_{x=x_L} f_2(t) \right] + \left[ h_2 \theta \Big|_{x=x_L} + h_2 \phi_1 \Big|_{x=x_L} f_1(t) + h_2 \phi_2 \Big|_{x=x_L} f_2(t) \right] = h_2 f_2(t) \end{array} \right.$$

$$\left\{ \begin{array}{l} -k \frac{\partial \theta}{\partial x} \Big|_{x=x_0} + h_1 \theta \Big|_{x=x_0} = 0 \\ k \frac{\partial \theta}{\partial x} \Big|_{x=x_L} + h_2 \theta \Big|_{x=x_L} = 0 \end{array} \right.$$

Step 3: Substitute  $T(x, t) = \theta(x, t) + \phi_1(x)f_1(t) + \phi_2(x)f_2(t)$   
into original initial condition. (with  $t=0$ )

$$\theta \Big|_{t=0} + \phi_1(x)f_1(0) + \phi_2(x)f_2(0) = F(x)$$

$$\theta \Big|_{t=0} = \underbrace{F(x) - [\phi_1(x)f_1(0) + \phi_2(x)f_2(0)]}_{F^*(x)}$$

Therefore: For  $\theta(x, t)$ ,

$\frac{\partial^2 \theta(x, t)}{\partial x^2} + g^*(x, t) = \frac{1}{\alpha} \frac{\partial \theta(x, t)}{\partial t}$	← nonhomogeneous equation
B.C. $\left\{ \begin{array}{l} -k \frac{\partial \theta}{\partial x} \Big _{x=x_0} + h_1 \theta \Big _{x=x_0} = 0 \\ k \frac{\partial \theta}{\partial x} \Big _{x=x_L} + h_2 \theta \Big _{x=x_L} = 0 \end{array} \right.$	← homogeneous B.C.
I.C. $\theta \Big _{t=0} = F^*(x)$	← homogeneous B.C.

$T(x, t)$  is obtained by solving for  $\theta(x, t)$ ,  $\phi_1(x)$  and  $\phi_2(x)$ .

## 2. Steady-state Conduction ( $\frac{\partial}{\partial t} = 0$ )

General Equation (3D):

$$\nabla^2 T(\vec{r}) + \frac{1}{k} g(\vec{r}) = 0$$

### 2.1. One-Dimensional Steady-state Conduction

#### 2.1.1. Rectangular Coordinate System - $T(x)$

\* Heat Conduction Equation:

$$\frac{d^2 T(x)}{dx^2} + \frac{1}{k} g(x) = 0$$

\* Special case:  $g(x) = g_0$  (constant heat generation)

Heat Conduction Equation:

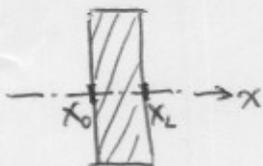
$$\frac{d^2 T(x)}{dx^2} + \frac{g_0}{k} = 0$$

General Solution:

$$\frac{dT(x)}{dx} = -\frac{g_0}{k}x + C_1 \quad (\text{1st integration})$$

$$T(x) = -\frac{g_0}{2k}x^2 + C_1x + C_2 \quad (\text{2nd integration})$$

$C_1$  and  $C_2$  are integration constants, determined by two boundary conditions.



## 2.1.2 Cylindrical Coordinate System — $T(r)$

\* Heat Conduction Equation:

$$\boxed{\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{1}{k} g(r) = 0}$$

\* Special Case:  $g(r) = g_0$  (constant heat generation)

Heat conduction Equation:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{g_0}{k} = 0$$

General Solution:

$$\frac{dT(r)}{dr} = -\frac{g_0}{2k} r + \frac{C_1}{r} \quad (\text{1st integration})$$

$$\underline{T(r) = -\frac{g_0}{4k} r^2 + C_1 \ln r + C_2} \quad (\text{2nd integration})$$

$C_1$  and  $C_2$  are integration constants, determined by two boundary conditions.

Note:

In the case of solid cylinder,

One boundary condition can be specified for the outer surface. Where is another boundary condition?



$$\underline{T|_{r=0} = \text{finite}} \quad (\text{solid cylinder})$$

Physically meaningful solution requires that temperature not be infinite at  $r=0$ .

## 2.1.3 Spherical Coordinate System - $T(r)$

\* Heat Conduction Equation

$$\boxed{\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) + \frac{1}{k} g(r) = 0}$$

\* Special Case:  $g(r) = g_0$  (constant heat generation)

Heat conduction Equation:

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) + \frac{g_0}{k} = 0$$

General solution:

$$\frac{dT(r)}{dr} = -\frac{g_0}{3k} r + \frac{C_1}{r^2} \quad (\text{1st integration})$$

$$T(r) = -\frac{g_0}{6k} r^2 - \frac{C_1}{r} + C_2 \quad (\text{2nd integration})$$

$C_1$  and  $C_2$  are integration constants, determined by two boundary conditions.

Note: In the case of solid sphere,

One boundary condition can be specified for the outer surface, where is another boundary condition?



$$\underline{T|_{r=0} = \text{finite}} \quad (\text{solid sphere})$$

Physically meaningful solution requires that temperature not be infinite at  $r=0$ .